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A New Method for Solving a Class of Ballot Problems

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1. INTRODUCTION

The classical ballot problem is to determine the probability that the winning candidate in a two-candidate election was never losing throughout the election. The problem has been attributed (cf. [5, 6, 9]) to Bertrand [2], more precisely attributed (cf. [1]) to Whitworth [13, 14], and perhaps even goes back (cf. [11]) to De Moivre [4]. An account of different methods for solving different variations on the classical ballot problem can be found in recent books by Mohanty [9] and Narayana [10]. The purpose of this paper is to demonstrate a new method for solving ballot problems which uses a certain formula for the derivative of a determinant to enumerate all possible ways in which the voting could have taken place.

First, we formulate what is meant by the classical ballot problem. Suppose A_1 and A_2 are two candidates, n is the total number of votes cast in the election, and $A_1(m)$ and $A_2(m)$ are the numbers of votes A_1 and A_2 have received, respectively, after the first m votes have been cast. Suppose further that $a_1 = A_1(n)$ and $a_2 = A_2(n)$ are given with $a_1 > a_2$ so that A_1 is the winner. Then the classical ballot problem is to determine the probability that

$$A_1(m) \geqslant A_2(m) \qquad \text{for} \quad m = 1,..., n-1. \tag{1}$$

The more general ballot problem which we shall consider here deals with k candidates $A_1,..., A_k$, in an election in which n votes have been cast. As in the classical problem, but for $i = 1,..., k$, let $A_i(m)$ denote the number of

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votes A_i received after the first m votes have been cast, and let $a_i = A_i(n)$. Suppose there are positive integers $t_1, \ldots, t_k$ such that

$$a_i < a_{i-1} + t_i \qquad \text{for} \quad i = 1, \ldots, k, \tag{2}$$

where we take $a_0 = a_k$ in the case $i = 1$. Indeed, to simplify notation throughout this paper, we fix the convention that

($*$) subscrits are to be considered mod k.

In this paper we shall determine the probability that

$$A_i(m) < A_{i-1}(m) + t_i \quad \text{for } i = 1, \ldots, k \text{ and } m = 1, \ldots, n-1. \tag{3}$$

Note that (3) is the same as (1) when $k = 2$, $t_1 > n$, and $t_2 = 1$.

We concern ourselves with the number $E = E(t_1, \ldots, t_k) = E(t_1, \ldots, t_k; a_1, \ldots, a_k)$ of ways of obtaining $A_i(n) = a_i$ for $i = 1, \ldots, k$ subject to the restrictions given by (2) and (3). The desired probability is then simply

$$E(t_1, \ldots, t_k; a_1, \ldots, a_k) \bigg/ \binom{n}{a_1, \ldots, a_k}, \tag{4}$$

where

$$\binom{n}{a_1, \ldots, a_k} = \frac{n!}{a_1! \cdots a_k!}.$$

For the calculation of E, we define $S(i, j) = \sum_{v=1}^{i} t_v - \sum_{v=1}^{j} t_v$ and $T = \sum_{v=1}^{k} t_v$. In Section 4 we shall prove the following

THEOREM. *For arbitrary positive integers $t_1, \ldots, t_k$, we have*

$$E(t_1, \ldots, t_k) = \sum_{\substack{u_1 = -\infty \\ u_1 + \cdots + u_k = 0}}^{\infty} \cdots \sum_{u_k = -\infty}^{\infty} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \binom{n}{b_{1\sigma(1)}, \ldots, b_{k\sigma(k)}}$$

$$= \sum_{\substack{u_1 = -\infty \\ u_1 + \cdots + u_k = 0}}^{\infty} \cdots \sum_{u_k = -\infty}^{\infty} n! \det((b_{ij}!)^{-1}),$$

where S_k is the symmetric group on k letters, $b_{ij} = a_j + u_i T + S(i, j)$, and we adhere to the convention here (and throughout this paper) that

($**$) *for any a and for any $m < 0$, $a/m! = 0$.*

Thus, the sums appearing in the Theorem may be viewed as finite sums. Also, the multinomial coefficients make sense since

$$\begin{aligned}\sum_{i=1}^{k} b_{i\sigma(i)} &= \sum_{i=1}^{k} (a_{\sigma(i)} + u_i T + S(i, \sigma(i)))\\ &= \sum_{i=1}^{k} a_i + \left(\sum_{i=1}^{k} u_i\right) T + \sum_{i=1}^{k} S(i, \sigma(i))\\ &= n + \sum_{i=1}^{k} \left(\sum_{v=1}^{i} t_v - \sum_{v=1}^{\sigma(i)} t_v\right) = n.\end{aligned}$$

Finally, we note that the last equality in the Theorem is trivial.

2. Applications

The general Theorem of Section 1 implies some results already known in the study of ballot problems. In this section we shall give some examples of such results and consider some further applications. We begin with a result which Barton and Mallows [1] show follows from a theorem of Karlin and McGregor [8].

Corollary 1. *The number N of ways of obtaining $A_i(n) = a_i$, for $i = 1,\ldots, k$, with the restrictions $A_i(m) < A_{i-1}(m) + t_i$, for $i = 2,\ldots, k$ and $m = 1,\ldots, n$, is*

$$N = n! \det((u(i, j)!)^{-1}),$$

where $u(i, j) = a_j + S(i, j)$.

Proof. Consider $t_1 > n$ in the Theorem. Then since $A_1(m) \leqslant a_1 \leqslant n < A_k(m) + t_1$ for $m = 1,\ldots, n$, we have $E(t_1,\ldots, t_n) = N$. The corollary will be proven if we can show that the only non-zero term appearing in the sum of the Theorem is the term with $u_1 = u_2 = \cdots = u_k = 0$. To show this consider any other term. Since $u_1 + \cdots + u_k = 0$, we must have some $u_i < 0$. For this choice of i, we have

$$\begin{aligned}a_j + u_i T + S(i, j) &\leqslant a_j - \sum_{v=1}^{k} t_v + \sum_{v=1}^{i} t_v - \sum_{v=1}^{j} t_v\\ &\leqslant a_j - \sum_{v=1}^{k} t_v + \sum_{v=1}^{k} t_v - t_1 < a_j - n \leqslant 0.\end{aligned}$$

By (**), the term under consideration is zero, completing the proof.

If in addition to taking $t_1 > n$ in the proof above we take $t_2 = t_3 = \cdots = t_k = 1$, we get, upon evaluating explicitly the resulting determinant in Corollary 1, a theorem of Thrall [12]. We also note that the following result, which can be found in [9] or [10], is a direct consequence of choosing $k = 2$ in the Theorem of Section 1.

COROLLARY 2. *The number N of ways of obtaining $A_i(n) = a_i$ for $i = 1$ and 2, with the restrictions $A_1(m) < A_2(m) + t_1$ and $A_2(m) < A_1(m) + t_2$ for $m = 1, \ldots, n$, is*

$$N = \sum_{u=-\infty}^{\infty} \left\{ \binom{a_1 + a_2}{a_1 + u(t_1 + t_2)} - \binom{a_1 + a_2}{a_2 + u(t_1 + t_2) + t_1} \right\},$$

where $\binom{a}{b} = 0$ if $b < 0$ or $b > a$.

Finally, we consider an application to Kolmogorov–Smirnov statistics. In our terminology, we define for $i, j = 1, \ldots, k$,

$$D_{i,j}^{+} = \max_{m=1,\ldots,n} \left(\frac{A_j(m)}{a_j} - \frac{A_i(m)}{a_i} \right).$$

When $n = kr$ and $a_1 = \cdots = a_k = r$, this gives

$$rD_{i,j}^{+} = \max_{m=1,\ldots,n} (A_j(m) - A_i(m)).$$

In the case $k = 2$ Gnedenko and Korolyuk [7] have found a formula for $P(\max(rD_{1,2}^{+}, rD_{2,1}^{+}) < t)$, where t is an arbitrary positive integer. In the case $k = 3$ David [3] found a formula for $P(\max(rD_{1,2}^{+}, rD_{2,3}^{+}, rD_{3,1}^{+}) < t)$. From (4) and our Theorem, we can generalize these probabilistic results to obtain

COROLLARY 3. *For arbitrary positive integers $t_1, \ldots, t_k$, and subject to convention (**), we have*

$$P(rD_{1,2}^{+} < t_2, rD_{2,3}^{+} < t_3, \ldots, rD_{k,1}^{+} < t_1)$$
$$= \frac{(r!)^k \, E(t_1, \ldots, t_k)}{(kr)!} = \sum_{\substack{u_1=-\infty \\ u_1 + \cdots + u_k = 0}}^{\infty} \cdots \sum_{u_k=-\infty}^{\infty} \det\left(\frac{r!}{b_{ij}!}\right),$$

where $b_{ij} = r + u_i T + S(i, j)$.

Similarly, large sample results corresponding to Kolmogorov–Smirnov statistics can be obtained. Using Stirling's formula, we have for any λ_i such that $\sum_{i=1}^{k} \lambda_i = 0$,

$$\lim_{r \to \infty} \frac{(r!)^k}{\prod_{i=1}^{k} (r + \lambda_i \sqrt{r})!} = \exp\left(-\frac{1}{2} \sum_{i=1}^{k} \lambda_i^2 \right).$$

Taking $t_i = s_i \sqrt{r}$ and using absolute convergence, we deduce

COROLLARY 4.

$$\lim_{r \to \infty} P(\sqrt{r}\, D_{1,2}^{+} < s_2, \ldots, \sqrt{r}\, D_{k,1}^{+} < s_1)$$
$$= \sum_{\substack{u_1 = -\infty \\ u_1 + \cdots + u_k = 0}}^{\infty} \cdots \sum_{u_k = -\infty}^{\infty} \sum_{\substack{\sigma \in S_k \\ \lambda_i \geq -\sqrt{r}}} (\operatorname{sgn} \sigma) \exp\left(-\frac{1}{2} \sum_{i=1}^{k} \lambda_i^2 \right),$$

where $\lambda_i = \lambda_i(\sigma) = u_i(\sum_{v=1}^{k} s_v) + (\sum_{v=1}^{i} s_v - \sum_{v=1}^{\sigma(i)} s_v)$ *and where the limit is over those* r *for which* $s_i\sqrt{r}$ *is integral for all* $i = 1, \ldots, k$.

3. The Classical Ballot Problem

In this section we shall illustrate our method by showing how the derivative of a determinant can be used to solve the classical ballot problem. In other words, we shall take $k = 2$ and, assuming $a_1 \geq a_2$, calculate the number $E = E(n+1, 1; a_1, a_2)$ of ways of obtaining $A_1(n) = a_1$ and $A_2(n) = a_2$ with the restriction $A_2(m) \leq A_1(m)$ for $m = 1, \ldots, n$. As we have remarked in Section 1, the probability of the winning candidate A_1 never being in a losing situation throughout the election is $E/\binom{n}{a_1}$. To calculate E, we consider a $k \times k$ matrix $A(x) = (a_{ij}(x))$ with differentiable elements and set $F(x) = \det A(x)$. Then

$$F'(x) = \sum_{\substack{l_1 + \cdots + l_k = 1 \\ l_i = 0 \text{ or } 1}} \det(a_{ij}^{(l_i)}(x)). \tag{5}$$

For the classical ballot problem we take $k = 2$ and

$$A(x) = \begin{bmatrix} x^{a_1}/a_1! & x^{a_2-1}/(a_2-1)! \\ x^{a_1+1}/(a_1+1)! & x^{a_2}/a_2! \end{bmatrix}.$$

Using (5) we successively calculate $F(x)$, $F'(x)$, $F''(x)$,..., $F^{(n)}(x)$ as a sum of non-zero determinants and arrive at a result which is easily checked by induction, namely,

$$F^{(m)}(x) = \sum_{\substack{d_1+d_2=m \\ d_1 \geqslant 0, d_2 \geqslant 0}} E(n+1, 1; d_1, d_2) \det A_{d_1,d_2}(x),$$

where, for $1 \leqslant i, j \leqslant 2$, the ijth element of $A_{d_1,d_2}(x)$ is given by $x^{a_i+i-j-d_i}/(a_i+i-j-d_i)!$ and where again we must keep in mind convention (**). In particular, the only non-zero term of $F^{(n)}(x)$ is when $d_1 = a_1$ and $d_2 = a_2$ so that

$$F^{(n)}(x) = E(n+1, 1; a_1, a_2) \det A_{a_1,a_2}(x).$$

But $A_{a_1,a_2}(x)$ is a lower triangular matrix with ones on the diagonal so that

$$F^{(n)}(x) = E(n+1, 1; a_1, a_2).$$

Now we calculate $F^{(n)}(x)$ directly. We have

$$F(x) = \det A(x) = \left(\frac{1}{a_1!\, a_2!} - \frac{1}{(a_1+1)!\,(a_2-1)!}\right) x^n$$

so that

$$F^{(n)}(x) = \binom{n}{a_1} - \binom{n}{a_1+1}.$$

Thus, we get

$$E(n+1, 1; a_1, a_2) = \binom{n}{a_1} - \binom{n}{a_1+1}$$

which, by our Theorem, is the result to be expected. Hence, the answer to the classical ballot problem is

$$\frac{\binom{n}{a_1} - \binom{n}{a_1+1}}{\binom{n}{a_1}} = \frac{a_1 - a_2 + 1}{a_1 + 1}.$$

4. The Proof of the Theorem

We shall require the following

Lemma. *Let $b_1, b_2, \ldots, b_k$ be non-negative integers and $\sigma \in S_k$ such that $b_i = a_{\sigma(i)} + u_i T + S(i, \sigma(i))$ for all $i = 1, \ldots, k$, and some integers u_i. Further suppose that $b_i < b_{i-1} + t_i$ for all i and that $\sum_{i=1}^k b_i = \sum_{i=1}^k a_i$. Then for all i we have $\sigma(i) = i$, $u_i = 0$, and hence $b_i = a_i$.*

Before proving the lemma we show how the theorem follows from it. The idea is to replace the $a_{ij}(x)$ of Section 3 with

$$a_{i,j}(x) = \sum_{u=-\infty}^{\infty} \frac{x^{a_j + uT + S(i,j)}}{(a_j + uT + S(i, j))!}.$$

Again, let $F(x) = \det A(x)$. Then

$$F^{(m)}(x) = \sum_{\substack{b_1 + \cdots + b_k = m \\ b_1, \ldots, b_k \geqslant 0}} E(t_1, \ldots, t_k; b_1, \ldots, b_k) \times \det\left(\sum_{u=-\infty}^{\infty} \frac{x^{a_j + uT + S(i,j) - b_i}}{(a_j + uT + S(i, j) - b_i)!} \right).$$

When $m = n$ and $x = 0$, $E(t_1, \ldots, t_k; b_1, \ldots, b_k)$ and the corresponding determinant are both non-zero precisely when there is a $\sigma \in S_k$ such that the conditions of the lemma hold. From the lemma, we see that such a determinant has value one and

$$F^{(n)}(0) = E(t_1, \ldots, t_k; a_1, \ldots, a_k).$$

We now compute $F^{(n)}(0)$ directly. We have

$$\begin{aligned}
F(x) &= \det A(x) \\
&= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \prod_{i=1}^{k} \left\{ \sum_{u=-\infty}^{\infty} \frac{x^{a_{\sigma(i)} + uT + S(i,\sigma(i))}}{(a_{\sigma(i)} + uT + S(i, \sigma(i)))!} \right\} \\
&= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \sum_{u_1=-\infty}^{\infty} \cdots \sum_{u_k=-\infty}^{\infty} \prod_{i=1}^{k} \frac{x^{a_{\sigma(i)} + u_i T + S(i,\sigma(i))}}{(a_{\sigma(i)} + u_i T + S(i, \sigma(i)))!} \\
&= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \sum_{u_1=-\infty}^{\infty} \cdots \sum_{u_k=-\infty}^{\infty} \frac{x^{\sum_{i=1}^k (a_{\sigma(i)} + u_i T + S(i,\sigma(i)))}}{\prod_{i=1}^k (a_{\sigma(i)} + u_i T + S(i, \sigma(i)))!}.
\end{aligned}$$

Since for any $\sigma \in S_k$ we have that $\sum_{i=1}^{k} a_{\sigma(i)} = n$ and $\sum_{i=1}^{k} S(i, \sigma(i)) = 0$,

$$F^{(n)}(x) = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \times \sum_{u_1=-\infty}^{\infty} \cdots \sum_{u_k=-\infty}^{\infty} \left\{ \frac{(n + \sum_{i=1}^{k} u_i)!\, x^{(\sum_{i=1}^{k} u_i) T}}{(\sum_{i=1}^{k} u_i)! \prod_{i=1}^{k} (a_{\sigma(i)} + u_i T + S(i, \sigma(i)))!} \right\}.$$

Now, taking $x = 0$, we have the conclusion of the Theorem.

Proof of Lemma. We first show that each u_i appearing in the lemma is ± 1 or 0. From the hypothesis, we have for $s = 0, 1, \ldots, k$, and for any integers i and j,

$$b_{i+s} - \sum_{v=i+1}^{i+s} t_v \leqslant b_i \leqslant b_{i+s} + \sum_{v=i+s+1}^{i+k} t_v \tag{6}$$

and

$$a_{j+s} - \sum_{v=j+1}^{j+s} t_v \leqslant a_j \leqslant a_{j+s} + \sum_{v=j+s+1}^{j+k} t_v. \tag{7}$$

In (6) and (7), equality holds in the inequalities on the left only for $s = 0$ and in the inequalities on the right only for $s = k$. Thus, for any j_1, j_2, i_1, and i_2,

$$|a_{j_1} - a_{j_2}| < T \qquad \text{and} \qquad |b_{i_1} - b_{i_2}| < T. \tag{8}$$

Now, if for some i there is a j such that $b_i = a_j + u_i T + S(i, j)$ with $u_i \geqslant 2$, then from (6) and (7), for $s = 0, 1, \ldots, k-1$,

$$\begin{aligned} b_{i+s} &> b_i - \sum_{v=i+s+1}^{i+k} t_v \geqslant \left(a_j + 2T + \sum_{v=1}^{i} t_v - \sum_{v=1}^{j} t_v \right) - \sum_{v=i+s+1}^{i+k} t_v \\ &= a_j + 2T - \sum_{v=i+s+1}^{j+k} t_v = a_j + \sum_{v=1}^{i+s} t_v + \sum_{v=j+1}^{k} t_v \\ &= a_j + \sum_{v=j+1}^{i+s+k} t_v \geqslant a_{i+s+k} = a_{i+s}. \end{aligned}$$

Therefore, $\sum_{i=1}^{k} b_i > \sum_{i=1}^{k} a_i$, giving a contradiction. Thus, we must have $u_i \leqslant 1$. Similarly, $u_i \geqslant -1$.

Now, we show that $u_i = 0$. Suppose that there is an i_1 between 1 and k inclusive such that $b_{i_1} = a_{\sigma(i_1)} + T + S(i_1, \sigma(i_1))$. Then

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i = \sum_{i=1}^{k} (a_{\sigma(i)} + u_i T + S(i, \sigma(i)))$$
$$= \sum_{i=1}^{k} (a_{\sigma(i)} + u_i T) = \sum_{i=1}^{k} a_i + \left(\sum_{i=1}^{k} u_i \right) T,$$

so that there is an i_2 between 1 and k inclusive with $b_{i_2} = a_{\sigma(i_2)} - T + S(i_2, \sigma(i_2))$. Also, if there is an i_2 between 1 and k inclusive such that $b_{i_2} = a_{\sigma(i_2)} - T + S(i_2, \sigma(i_2))$, then there is an i_1 between 1 and k inclusive such that $b_{i_1} = a_{\sigma(i_1)} + T + S(i_1, \sigma(i_1))$.

If $i_1 \geqslant j_1 = \sigma(i_1)$, then by (6) and (7), for $s = 0, \ldots, k-1$,

$$b_{i_1+s} > b_{i_1} - \sum_{v=i_1+s+1}^{i_1+k} t_v = \left(a_{j_1} + T + \sum_{v=1}^{i_1} t_v - \sum_{v=1}^{j_1} t_v \right) - \sum_{v=i_1+s+1}^{i_1+k} t_v$$
$$= a_{j_1} + \sum_{v=j_1+1}^{i_1} t_v + \sum_{v=i_1+1}^{i_1+s} t_v \geqslant a_{i_1+s}.$$

Thus, $\sum_{i=1}^{k} b_i > \sum_{i=1}^{k} a_i$, giving a contradiction. So $i_1 < j_1$. Similarly, $i_2 > j_2 = \sigma(i_2)$.

Now, suppose $i_1 \geqslant i_2$. Thus, by (6),

$$a_{j_1} + T - \sum_{v=i_1+1}^{j_1} t_v = b_{i_1} \leqslant b_{i_2} + \sum_{v=i_2+1}^{i_1} t_v$$
$$= a_{j_2} - T + \sum_{v=j_2+1}^{i_2} t_v + \sum_{v=i_2+1}^{i_1} t_v$$

so that $a_{j_2} > a_{j_1} + T$, contradicting (8). On the other hand, if $i_1 < i_2$, then by (6),

$$a_{j_1} + T - \sum_{v=i_1+1}^{j_1} t_v = b_{i_1} < b_{i_2} + \sum_{v=i_2+1}^{k} t_v + \sum_{v=1}^{i_1} t_v$$
$$= a_{j_2} - T + \sum_{v=j_2+1}^{i_2} t_v + \sum_{v=i_2+1}^{k} t_v + \sum_{v=1}^{i_1} t_v$$

so that by (7),

$$a_{j_1} < a_{j_2} - \sum_{v=j_1+1}^{k} t_v - \sum_{v=1}^{j_2} t_v < a_{j_1},$$

giving a contradiction.

Therefore, $u_i = 0$, for all i, and to finish the proof, it suffices to show that if $b_i = a_{j_1} + S(i, j_1)$ and $b_{i+1} = a_{j_2} + S(i+1, j_2)$, with $1 \leqslant i, i+1, j_1, j_2 \leqslant k$, then $j_2 > j_1$. To see this, suppose $j_1 \geqslant j_2$. Then

$$a_{j_2} + S(i+1, j_2) = b_{i+1} < b_i + t_{i+1} = a_{j_1} + S(i, j_1) + t_{i+1},$$

so by (7),

$$\begin{aligned} a_{j_2} &< a_{j_1} - \sum_{v=1}^{i+1} t_v + \sum_{v=1}^{j_2} t_v + \sum_{v=1}^{i} t_v - \sum_{v=1}^{j_1} t_v + t_{i+1} \\ &= a_{j_1} - \sum_{v=j_2+1}^{j_1} t_v \leqslant a_{j_2}, \end{aligned}$$

leading to a contradiction and hence the conclusion of the proof.

Finally, we note that the method of the present paper can be adapted to the case when votes are counted in batches of two. The author plans to return to this and to some related questions in a subsequent paper.

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